Notes on the Birkhoff Polytope and Poset Polytopes

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Abstract

In this note, we summarize some of the current results on the Ehrhart polynomial of the Birkhoff polytope and poset polytopes, then investigate possible approaches to attack some main problems regarding these two polytope families. We also introduce the Birkhoff-von Neumann decomposition and provide code in SageMath that generates the snake posets.

1 Introduction

Polytopes are geometric objects living in Euclidean space of any dimensions, often described as the convex hull of a set of points or the intersection of half-spaces, and it's worth noting that it's not easy to prove the equivalence of two definitions and each applies to different problems. We call a convex polytope P in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$ a lattice polytope. Polytopes have been widely studied in discrete geometry and algebraic combinatorics, they arise on multiple occasions, due to various reasons and link different parts of mathematics, the two types of polytopes we are going to introduce appear frequently in other fields.

The Birkhoff polytope is usually known under the name *magic square* or *semi-magic square*, which dates back thousands of years ago. We denote $\mathcal{B}_n = \{A|A \in \mathbb{R}^{n\times n}, a_{ij} \geq 0, \sum_{i=0}^{n} a_{ij} = n, \sum_{j=0}^{n} a_{ij} = n\}$, all non-negative matrices whose rows and columns sum up to n. When viewed as a subset of $\mathbb{R}^{n^2}$, it's shown that $\mathcal{B}_n$ is a $(n-1)^2$-dimensional polytope with all $n \times n$ permutation matrices as its vertices, namely the convex hull of S_n (all permutation matrices). This leads us to the natural question of how to efficiently decompose $A \in \mathcal{B}_n$ as a linear combination of permutation matrices and analyze its run-time, and how hard this problem is, which will be addressed later in the note.

The poset polytopes we discuss in this note are introduced by Stanley in [7]. Given a poset $(P, \prec)$ with cardinality n, we define $\mathcal{O}(P) = \{f : P \to R | x_i \prec x_j \implies f(x_i) \leq f(x_j), 0 \leq f(x_i) \leq 1\}$, which can be naturally identified as a subset of $\mathbb{R}^n$ in the following way: $P = \{\alpha_1, \cdots, \alpha_n\}$, $\mathcal{O}(P) = \{(y_1, \cdots, y_n) : 0 \leq y_j \leq 1, y_i \leq y_j \text{ if } \alpha_i \prec \alpha_j\}$. It's a convex polytope because it's clearly bounded and defined by a finite number of linear inequalities, this is called the *order polytope*. Given one linear extension of P, $P = \{x_1, \cdots, x_n\}$, $\mathcal{O}(P)$ contains the simplex $\{p = (p_i) \in \mathbb{R}^n | p_1 \leq \cdots \leq p_n\}$, thus we know that $dim(\mathcal{O}(P)) = n$, which is an important distinction from $\mathcal{B}_n$.

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The order polytope has a close relative, the *chain polytope*. It's all functions $g : P \to R$ whose values are non-negative and $g(y_1) + \cdots + g(y_n) \leq 1$, for every chain $y_1 \prec \cdots y_n$. This is also an n-dimensional convex polytope denoted $\mathcal{C}(P)$.

There are many more polytopes arising from posets, another example is the *valuation polytope*, but we won't discuss this one in detail. Assume that L is a finite distributive lattice, a mapping $v : L \to \mathbb{R}$ is called a valuation if $v(a \wedge b) + v(a \vee b) = v(a) + v(b), \forall a, b \in L$. All valuations identify naturally to a point in $\mathbb{R}^n$ and we denote $V(L)$ to be all valuations with range $[0, 1]$, there are different definitions that are equivalent to this up to translation.

We'll study these polytopes under the framework of Ehrhart theory, which links lattice points enumeration with polynomials, see [3] for background information. Along the way, we present different techniques used to derive the Ehrhart polynomial and pay extra attention to the study of roots.

2 Background

2.1 Norm bound of roots of Ehrhart polynomials

For a convex lattice polytope P in $\mathbb{R}^d$, we denote its Ehrhart polynomial $L_P(t)$, we introduce some results on the roots of $L_P(t)$, denoted as a set R_P and R_P is bounded by c means all roots are bounded by c in usual Euclidean norm.

In [2], the authors found that R_P is bounded by $1 + (d+1)!$, and later in [4] B. Braun improved the bound tremendously to $\theta(d^2)$ using a rather simple argument by changing the basis of $L_P(t)$ and using known facts on the h^*-vector of P.

Theorem 1 (Theorem 1 in [4]). *If f is a non-zero polynomial of degree d with real-valued, non-negative coefficients when expressed with respect to the basis of polynomials of degree d*

$$V_d(t) = \left\{ \binom{t+d-j}{d} | 0 \leq j \leq d \right\}$$

then all roots of f lie inside the disc with center $-\frac{1}{2}$ and radius $d(d - \frac{1}{2})$.

If we take f to be $L_P(t)$, one can verify that it satisfies the assumption, since the coefficients of $L_P(t)$ under $\{V_d\}$ form the so-called the h^*-vector of P, which is proven to be a non-negative integer-valued vector.

We sketch the proof steps below:

1. view $f(z)$ as a linear combination of points $V_d(z)$ in $\mathbb{C}$, define $D_d = \{z : |z + \frac{1}{2}| \leq d(d - \frac{1}{2})\}$, take any $z \notin D_d$.

2. $M = \{(z+d), \cdots, (z-d+1)\}$, and show that the angular width of M is less than $\frac{\pi}{d}$.

3. show all $V_d(z)$ live in an open half-plane since they're all products of $\frac{1}{d!}$ and d elements in M, and since the coefficients are non-negative, the combination $f(z)$ is not zero, thus roots all live in D_d.

Remark 1. this proof only requires the coefficients to be non-negative, i.e. only depends on f having a "nice" representation with respect to V_d, if the coefficients satisfy extra properties such as unimodality we may improve the bound. The proof shows how a geometric view like viewing $f(z)$ as a linear combination of points can lead to a concise and intuitive proof.

2.2 Birkhoff Polytopes

We denote $L_{\mathcal{B}_n}(t) = H_n(t)$, and present a reciprocity/symmetry of $H_n(t)$.

Proposition 1 (Theorem 6.3 in [3])**.**

$$(-1)^{(n-1)^2} H_n(t) = L_{\mathcal{B}_n^\circ}(-t) = H_n(-t-n)$$

and

$$H_n(-1) = \cdots = H_n(-n+1) = 0$$

This result gives us a way to actually compute $H_n(t)$ by interpolation. We need to compute $(n-1)^2+1$ to interpolate, and n of them is known, by Ehrhart–Macdonald reciprocity we need to compute $((n-1)^2+1-n)/2 = \binom{n-1}{2}$ values, which is clearly too expensive. But it's fun that we can do H_3 by hand knowing that $H_3(1) = 6, H_3(0) = H_3(-3) = 1, H_3(-1) = H_3(-2) = 0$.

There are other ways to describe $\mathcal{B}_n$, for example:

Observation 1. Permutation matrices are in bijection with matchings on the complete bipartite graph $K_{n,n}$.

But a more concise way is the following:

$$\mathcal{B}_n = \{\mathbf{x} \in \mathbb{R}_{\geq 0}^{n^2} : A\mathbf{x} = \mathbf{b}\}$$

in which

$$A = \begin{pmatrix} 1 & \cdots & 1 & & & & & & & \\ & & & 1 & \cdots & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 1 & \cdots & 1 \\ 1 & & & 1 & & & & 1 & & \\ & \ddots & & & \ddots & & \cdots & & \ddots & \\ & & 1 & & & 1 & & & & 1 \end{pmatrix} \in \mathbb{Z}^{2n \times n^2}$$

and

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{Z}^{2n}$$

Remark 2. A is the incidence matrix of $K_{n,n}$.

2.3 Poset polytopes

We've defined $\mathcal{O}(P)$ and derived that $dim(\mathcal{O}(P)) = n$, next we follow [7] discussing its vertices, faces, volumes, etc.

a facet of a polytope is one of the maximal faces besides the entire polytope. We first characterize facets and vertices of $\mathcal{O}(P)$, and refer the reader to [7] for a full description of faces.

$f \in \mathcal{O}(P)$ is in a facet iff it satisfies exactly one of the following conditions:

1. $f(x) = 0$, x is a minimal element.

2. $f(x) = 1$, x is a maximal element.

3. $f(x) = f(y)$, y covers x.

thus the number of facets is $a + b + C(P)$ in which $a, b, C(P)$ are the number of minimal elements, maximal elements, and covering relations, respectively.

Define a subset $I \subset P$ to be a *filter* if: $\forall x \in I, \forall y \geq x$, we have $y \in I$, namely upward-closed.

Proposition 2. *$V(\mathcal{O}(P))$ consists of characteristic functions of filters I. In particular, the number of vertices is the number of filters.*

There's a similar characterization of the vertices of $\mathcal{C}(P)$, they are the characteristic functions of all *anti-chains*, and there's a well-known bijection between filters I and anti-chains J:

$$I = \{y : x \prec y, \exists x \in J\}$$
$$J = \text{minimal elements of } I$$

So $\mathcal{O}(P)$ and $\mathcal{C}(P)$ have the same number of vertices. In fact, there's a nice bijection between them and they have the same Ehrhart polynomial. There's a beautiful relation between $L_{\mathcal{O}(P)}(t)$, $L_{\mathcal{C}(P)}(t)$, and the *order polynomial* $\Omega(P, t)$, the number of order-preserving maps: $P \to [t]$.

Theorem 2. $L_{\mathcal{O}(P)}(t) = L_{\mathcal{C}(P)}(t) = \Omega(P, t + 1)$

Proof. By definition, $L_{\mathcal{O}(P)}(t)$ is the number of order preserving map: $P \to [0, 1]$ such that $tf(x) \in \mathbb{Z}, \forall x \in P$, which equals to the number of order preserving map: $P \to \{0, 1, \cdots, t\}$, which trivial equals to number of order preserving map: $P \to \{1, \cdots, t + 1\}$. Thus $L_{\mathcal{O}(P)}(t) = \Omega(P, t + 1)$. □

Compare the leading coefficients of the two and we get a relation between the geometry of $\mathcal{O}(P)$ and the combinatoric nature of P.

Proposition 3. *Volume of $\mathcal{O}(p)$ equals to $\frac{e(P)}{n!}$, where $e(P)$ is the number of linear extensions.*

Stanley observed that $\mathcal{C}(P)$ is the *vertex packing* poltope of the comparability graph of P, combined with the above theorem.

Observation 2. The order polynomial of a finite poset P depends only on the comparability graph.

Because the comparability graph remains unchanged if we take the dual of a poset, we note one simple consequence.

Proposition 4. *The dual of P gives the same Ehrhart polynomial as that of P.*

Stanley also gives a canonical triangulation of $\mathcal{O}(P)$ in [7] using *order ideal* which is just **downward-closed** version of the filter.

Let $J(P)$ denote the lattice of order ideals ordered by inclusion, for any chain $K : I_1 \subset \cdots \subset I_k$, we can build a subset of $\mathcal{O}(P)$ by restricting f to be constant on $I_1, I_2 - I_1, \cdots, P - I_k$, and of course order-preserving. In particular, the facets are in bijection with all linear extensions:

$$0 = f(y_1) \leq \cdots \leq f(y_n) = 1$$

Remark 3 ($\mathcal{B}_n$, $\mathcal{O}(P)$, and $\mathcal{C}(P)$ are closely related!)**. Both $\mathcal{O}(P)$ and $\mathcal{B}_n$ are $0-1$ compressed polytopes.**

An $0-1$ polytope simply means that vertices are in $\{(i_1, \cdots, i_n) : i_j \in \{0,1\}\}$. By the main theorem in [1], we know that the h^*-vector of an $0-1$ compressed polytope shows unimodality. Moreover, both $\mathcal{O}(P)$ and $\mathcal{B}_n$ have symmetric root distribution, and we should expect more similarity such as a reciprocity for $\mathcal{O}(P)$.

But in terms of root distribution, $\mathcal{O}(P)$ tends to have more real roots and $\mathcal{B}_n$ has mostly roots with a non-zero imaginary part.

3 Several formulas of $H_n(t)$

Now we switch our attention to $H_n(t)$ and focus on methodology instead of concrete results.

3.1 Euler's generating function and Constant term identities

For any polytope given by

$$P = \{\mathbf{x} \in \mathbb{R}^{d^2}_{\geq 0} : A\mathbf{x} = \mathbf{b}\}$$

Denote all columns of A by $\mathbf{c}_1, \cdots, \mathbf{c}_d$, consider the constant term of the Euler's generating function:

$$\frac{1}{(1-\mathbf{z}^{\mathbf{c_1}})\cdots(1-\mathbf{z}^{\mathbf{c_d}})\mathbf{z}^{t\mathbf{b}}}$$

by expanding it in terms of geometric series:

$$(\sum \mathbf{z}^{n_1\mathbf{c_1}})\cdots(\sum \mathbf{z}^{n_d\mathbf{c_d}})\mathbf{z}^{-t\mathbf{b}}$$

for the constant term, we have:

$$n_1\mathbf{c_1} + \cdots + n_d\mathbf{c_d} - t\mathbf{b} = A\mathbf{n} - t\mathbf{b}$$

in which all n_i, t are non-negative integers. Thus we have the Constant term identity:

$$L_P(t) = const\left(\frac{1}{(1-\mathbf{z}^{\mathbf{c_1}})\cdots(1-\mathbf{z}^{\mathbf{c_d}})\mathbf{z}^{t\mathbf{b}}}\right)$$

from this we can plug in the specific $A, \mathbf{b}$ and get one formula of $H_n(t)$.

3.2 A multivariate generating function approach

See [5] for reference of this section. This section briefly introduces the use of Brion's theorem, triangulation, and Barvinok's algorithm.

Definition 3.1. the multivariate generating function of a pointed polyhedron P:

$$f(P, \mathbf{z}) = \sum_{M \in P \cap \mathbb{Z}^{n^2}} \mathbf{z}^M$$

where $\mathbf{z}^M = \prod_{1\le i,j\le n} z_{i,j}^{m_{i,j}}$. In particular,

$$f(t\mathcal{B}_n, \mathbf{z}) = \sum_{M \in t\mathcal{B}_n \cap \mathbb{Z}^{n^2}} \mathbf{z}^M$$

this clearly encodes more information than $H_n(t)$, since by plugging $z_{i,j} = 1$, we get $H_n(t)$.

A cone is a set of non-negative linear combinations of a finite set of vectors, given a cone C, we define the dual cone to be $C^* = \{y :< y, x >\ge 0, \forall x \in C\}$.

Remark 4. if P is not pointed, i.e. P contains a straight line, then MGF of P is zero. Also, it's proven that *in* $\mathbb{R}^n$, *dual of cones whose dimension is lower than* n *is not pointed*, thus they have zero MGF.

The authors derive a complicated but purely combinatorial formula for $f(t\mathcal{B}_n, \mathbf{z})$, which is not shown in detail here.

Why do we care about cones? See the following theorems.

Theorem 3 (Brion, 1988; Lawrence, 1991)**.** *Let* $S(P, v)$ *be the supporting cone of* P *at* v*.*

$$f(P, \mathbf{z}) = \sum_{v \in V(P)} f(S(P, v), \mathbf{z})$$

Algorithm 1. *Dual Barvinok algorithm*

Input: a rational full-dimensional cone C*.*

Output: the MGF of C*.*

1. *Find the dual* D *of* C*.*
2. *Triangulate* D *into simplicial cones, discarding lower-dimensional cones.*
3. *Apply Barvinok's signed decomposition to the simplicial cones until all are unimodular cones, discarding lower-dimensional cones.*
4. *Take the dual back and get a decomposition of* C*.*
5. *Summing over the MGFs of all these unimodular cones to get* $f(C, \mathbf{z})$*.*

This algorithm using *dual* and then pulling back is faster than the original algorithm in magnitude, mainly because we can discard all lower dimensional cones according to the remark.

4 Birkhoff-von Neumann decomposition

The main reference of this section is [6] and Wikipedia.

Given an $n \times n$ semi-magic square $A \in \mathcal{B}_1$, recall that A can be decomposed as $A = \alpha_1 P_1 + \cdots + \alpha_k P_k, \alpha_i \in (0, 1)$and $\sum_{i=0}^{k} \alpha_i = 1$, P_i's are permutation matrices, this is called a BvN decomposition. We'll partially answer the following questions:

- is BvN unique?

- given A, can we determine what's the minimum k?
- the computational complexity of the minimum-decision problem.

BvN is not unique in general, we illustrate that uniqueness fails even for a small matrix:

Example.

$$\begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}$$
$$= 0.2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 0.2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 0.3\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + 0.1\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 0.2\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$= 0.2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + 0.4\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 0.3\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 0.1\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Definition 4.1. A few concepts from *computational complexity theory.*

- a *decision problem* takes in any valid input and outputs "yes" or "no".
- P is a set of problems that can be solved by a deterministic Turing machine in Polynomial-time.
- NP is a set of decision problems that can be solved by a Non-deterministic Turing Machine in Polynomial-time. A problem is in NP if you can quickly (in polynomial time) test whether a solution is correct (without worrying about how hard it might be to find the solution).
- A problem L is NP-complete if: $L \in NP$ and every problem in NP is reducible to L in polynomial time.

Remark 5. the famous $P = NP$ basically says that if any solution to a problem can be **checked** within polynomial time, then a solution can be **found** in polynomial time. Many computer scientists believe that $P \neq NP$.

Theorem 4 (1 in [6])**.** *The problem of deciding whether there is a BvN of a given A with k permutation matrices is NP-complete.*

The proof relies on one of the most important ideas in complexity theory defined below.

Definition 4.2. a *reduction* is an algorithm for transforming one problem into another problem. A sufficiently efficient reduction from one problem to another may be used to show that the second problem is at least as difficult as the first.

Observation 3. P and NP are closed under polynomial-time reductions, namely if $A \in NP$ and A reduces to B in polynomial-time, then $B \in NP$. In other words, we preprocess A to make it B, thus solving B is at least as hard as solving A. This is the standard way to prove one problem to be in P or NP, namely if $A \in NP$, show that if we can solve an instance of A by solving an instance of B, then $B \in NP$.

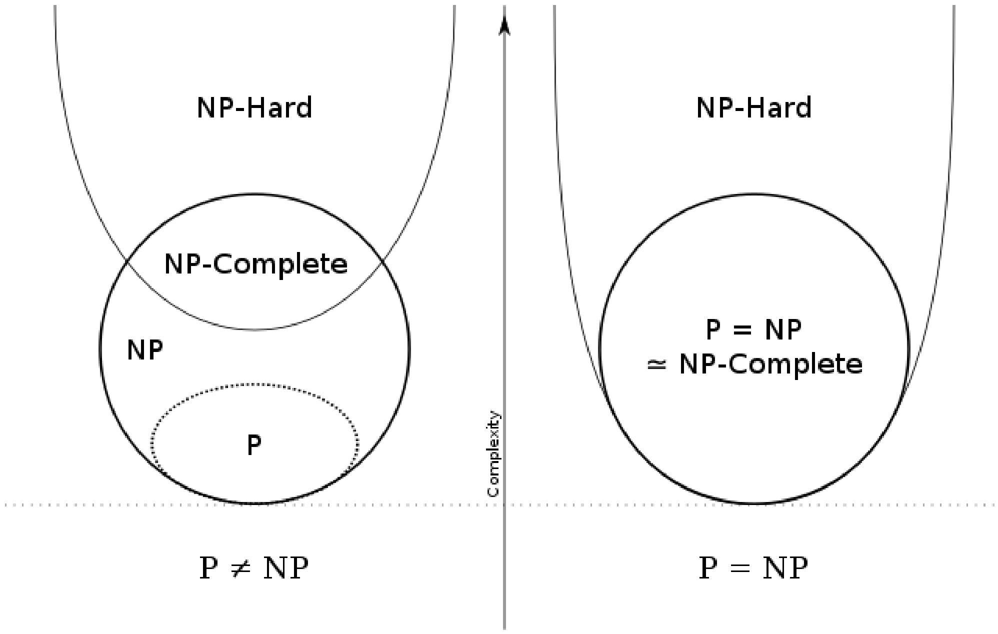

Figure 1: Classes of computational problems

The proof of the NP-completeness of BvN is just as we outlined. First, it's clear that it's in NP since it's easy to verify whether k permutation matrices sum up to A or not. Then take a 3-partition problem and build a matrix and a number (k) out of it, show that finding a BvN with k permutation matrices is equivalent to finding a 3-partition, and lastly, use the fact that the 3-partition is an NP problem.

The way to deal with an NP-complete problem practically is by using heuristics, a greedy heuristic is described in [6] and happens to give us the second BvN in the example showing non-uniqueness.

5 Computational approach for poset polytopes

See the reference for the so-called **snake poset** in [10] and [9]. I wrote code in SageMath that can generate **Snake** object and methods that allow us to see the Ehrhart polynomials of all snake posets of order n, you can see and download the code here.

This allows us to experiment and find patterns, for example, one observed property of snake poset is that a pair of palindromic words give us two snake posets with the same Ehrhart polynomial, e.g. these four words $\{RLLL, LRRR, LLLR, RRRL\}$ produce the same Ehrhart polynomials. It's not surprising that $(RLLL, LRRR)$ produce the same Ehrhart polynomials since the corresponding posets are isomorphic, and a little bit of thought tells us that palindromic words give a pair of dual snakes and thus have the same Ehrhart polynomials.

Another observed phenomenon concerning the roots of the Ehrhart polynomial of snake posets is described in the following section in detail.

6 Conjectures and Possible approaches

Below we list some conjectures and open problems and propose several possible approaches.
Conjectures:

The first two conjectures are based on the experimental data of the roots distribution of $H_9(t)$ from [2], see figure 2.

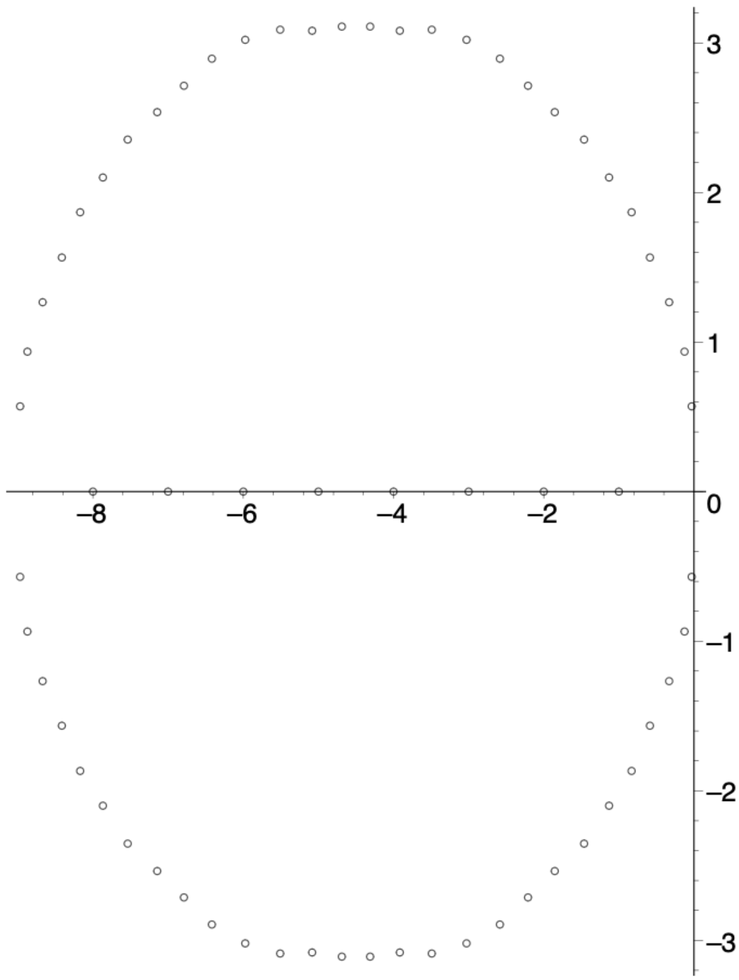

Figure 2: the roots distribution of $H_9(t)$

- the norm of the roots of $H_n(t)$ is bounded by $O(\sqrt{n})$, better than the general bound $O(n^2)$.
- all roots of $H_n(t)$ have negative real parts and thus $H_n(t)$ exhibits Ehrhart positivity.
- Stanley gives a conjectured asymptotic behavior of coefficients of $H_n(t) = \sum_{i=0}^{(n-1)^2} b(n,i)t^{(n-1)^2-i}$ in [8], the exercise 4.54, as follows:
$$\frac{b(n,k)}{b(n,0)} \sim \frac{n^{3k}}{2^k k!}$$
as $n \to \infty$.

Also based on experiments using snake generation, we have the following observations/speculations for any snake word with length n (the initial one is not counted), we denote its Ehrhart polynomial $ES_n(t), n \geq 0$:

- $ES_n(t)$ shows Ehrhart positivity.
- roots of $ES_n(t)$ distribute symmetrically with respect to the line $x = \frac{-n-4}{2}$. Moreover, there should be reciprocity.
- if the word is "$RR\cdots R$" or "$LL\cdots L$", then the roots are $(-n-3,1), (-n-2,2), (-n-1,2), \cdots (-2,2), (-1,1)$, the first entry in the tuple is a root and the second is its multiplicity. In particular, all roots of the straight snake are real. Maybe one can solve this by known results in flow polytopes.

And some approaches or directions to explore:

- in [5], the authors use *multivariate generating function of lattice points* (MGF) to derive the Ehrhart polynomial of the Birkhoff polytope. The algorithm to get the MGF relies on Brion's theorem, Brion's polarization trick, and Barvinok's algorithm. It's possible to apply the same algorithm to $\mathcal{O}(P)$ which requires us to study its supporting cones and dual cones. Using Brion's formula and Theorem 3.5 in [3] is a similar promising way.
- the approach in [4] only uses the non-negativity of h*-vector, it's proven that the h*-vector of the Birkhoff polytope is unimodal and may be used to improve the bound using the same idea in [4]. Can we use a different basis? such as consider $\sum_{n\geq 1} \frac{L_P(t)}{n!}x^n$.
- finding correspondence between posets and graphs, rephrasing some problems in graph theory language, borrowing techniques, and known results from graph theory. E.g. Stanley's observation that *"the order polynomial of a finite poset P depends only on the comparability graph"* relies on rephrasing $\mathcal{C}(P)$ as an object built purely from the graph of P.

The following part also comes from the talk given by Anastasia Chavez at UC Berkeley on May 1st.

- how do poset operations affect $\mathcal{O}(P)$, $V(L)$ and their Ehrhart polynomials? E.g.which operation on a poset corresponds to adding and which corresponds to multiplying, what does ordinal sum correspond to, etc.

- Once we have results on the first question, can we decompose a poset in some way to compute its Ehrhart polynomial and many other attributes?

  This seems to be a natural step towards generality given how decomposition is ubiquitous in math: decomposing an event E using conditional probability to compute $P(E)$, decomposing a topological space to simple parts to compute the fundamental group of it using Van Kampen theorem, etc. Triangulation is also one kind of decomposition but it seems to be easier to decompose at the poset level.

  Another piece of evidence supporting this is that all faces of $\mathcal{O}(P)$ are given by some partitions of P, thus this could be parallel to computing the Ehrhart polynomial of a polytope via its facets/faces.

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