Repeated Games and Folk Theorem

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1 Game models

We first introduce a game model and establish the corresponding terminology, we'll see how definitions and names enable us to **formalize** a *game* to a familiar and rigorous mathematical object.

Definition 1.1 (language we need to talk about a game)**.**

- A game is a triple $(N, (S^i)_{i\in N}, (g_i)_{i\in N})$, $g_i : S = \prod_{i\in N} S^i \to \mathbb{R}$ is called the *payoff* function of player i. We may use I and N interchangeably to denote all players.
- a *strategy profile* is an element $s = (s^i)$ in S, $s^{-i} = \prod_{j\neq i} s^j$, similar define $S^{-i} = \prod_{j\neq i} S^j$.
- $\Delta(S^i) = \{\sigma : \sigma$ is a probability distribution on $S^i\}$, the mixed extension of G is $(N, (\Delta(S^i))_{i\in N}, (\tilde{g}_i)_{i\in N})$, where $\tilde{g}_i(\sigma) = \sum_{a\in S^i} \sigma(a)g_i(a)$.

1.1 Nash equilibrium

Definition 1.2. A strategy profile s is said to be a *Nash equilibrium* of G if:

$$\forall t^i \in S^i, g_i(s^i) \geq g_i(t^i, s^{-i})$$

i.e. unilateral deviation won't result in more profits, thus no one has the incentive to change his strategy.

The Nash equilibrium is a fixed point if we add in proper topology and geometry - in this case, convexity! Thus we can use Brouwer's fixed point theorem or Kakutani's fixed point theorem to prove the following theorems. Next, we establish several scenarios that admit the existence of Nash equilibrium. The result in the finite case is satisfactory. When we say G is a finite game we mean there are a finite number of players and each of their strategy sets is finite.

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Theorem 1 (Nash Equilibrium in Finite Games). *Every finite game G has a mixed equilibrium. A mixed equilibrium of G is a Nash equilibrium of $\tilde{G}$.*

To prove the existence of Nash equilibria in a continuous setting, we need to assume some topological and geometrical conditions. The game is called compact if each strategy set S^i is a compact subset of a topological space and each payoff function g_i is bounded. Finally, the game G is called continuous if each S^i is a topological space and each payoff function g_i is continuous with respect to the product topology on S. We will assume that each strategy set S^i is a convex subset of a topological vector space, and correspondingly a geometric condition on g_i, a real-valued function f on a convex set X is quasi-concave if the level sets $\{x : f(x) \geq \alpha\}$ are convex for all reals α.

Definition 1.3. A game is quasi-concave if for all $i \in N$ the map $s^i \to g(s^i, s^{-i})$ is quasi-concave for all s^{-i} in S^{-i} and all $i \in N$.

Theorem 2. *If a game G is compact, continuous and quasi-concave, then its set of Nash equilibria is a non-empty and compact subset of $\prod_{i \in N} S^i$.*

1.2 repeated games

Example (Prisoner's dilemma). Consider the following game 1 in which the rows are strategies of the first player and the first coordinate of a pair is the payoff of player 1.

In the mixed extension of this game, the Nash equilibrium is unique, namely $(1, 1)$.

What if we repeat the game for finite times, will the prisoners realize that collaboration is the better solution? No, they won't! Repeat the game for n times and consider the average payoffs, denote the E_T the equilibrium payoffs up to round T, we can prove by induction that $E_T = \{(1, 1)\}$.

Take an $T + 1$ stage equilibrium profile σ, assume player 1 uses $C1$ with probability x and $B1$ with y, in the first round under σ, the profit will be greater or equal to than the case in which player 1 uses $B1$ in the first round, notice that the continuation strategies induced by σ form a Nash equilibrium of the remaining game.

$$\frac{1}{1+T}(3xy + 4(1-x)y + (1-x)(1-y) + T) \geq \frac{1}{1+T}(4y + (1-y) + T)$$

solving this we obtain: $x = 0$, similarly $y = 0$, and we proved that $E_{T+1} = \{(1, 1)\}$. takeaway: there maybe be no room for cooperation in a finite repeated game.

	C2	B2
C1	(3, 3)	(4, 0)
B1	(0, 4)	(1, 1)

Table 1: Prisoner's dilemma

Definition 1.4. We define a few fundamental concepts when talking about repeated games.

- a history of length t is a vector $(a_1, \cdots, a_t)$, $a_i \in S$ is the strategy profile at stage t, let H_t be all histories of length t and $H = \bigcup_{t \geq 0} H_t, H_0 = \varnothing$.

- a *strategy* of player i is a map $\sigma^i = (\sigma^i_j)_{j\geq 1}$: $H \to \Delta(S^i)$, a profile is defined same as before. Σ_i is the strategy set of i and $\Sigma = \prod_{i\in N} \Sigma_i$. These are pure strategies! Even they map into distributions.

- $u = (a_1, a_2, \cdots)$ is called a *path* or a *play*.

This is how to interpret the definition: σ^i takes in history up to stage t and determines the probability distribution on S^i that player i should adopt at stage $t+1$. A *profile* $\sigma = (\sigma^i)_{i\in N}$ defines a probability distribution on H inductively, first use $(\sigma^i_1)_{i\in N}$ as the distribution on H_1, then extended by Kolmogorov's theorem to H_∞.

A Nash equilibrium for a repeated game means that no player can deviate on any round unilaterally to gain profits, but how do we define payoff here? Or more fundamentally, what are the characteristics we are modeling when formulating different kinds of payoffs? One such thing is the patience of the players, they can be gambling and really only care about how much they win in an hour, or they could be in a large-scale game and are patient enough to value all stages equally.

Definition 1.5. Given a profile σ.

- (finitely repeated game) the payoff of i up to stage T, $\gamma^i_T(\sigma) = \mathbb{E}_\sigma(\frac{1}{T}\sum_{t=1}^{T} g^i(a_t))$. And $\frac{1}{T}\sum_{t=1}^{T} g^i(a_t)$ is indeed a random variable on H, it can also be viewed as an r.v. on $\bigcup_{t=0}^{T} H_t$ and the measure is simply defined by induction without using Kolmogorov extension.

- (discounted game) $\gamma^i_T(\sigma) = \mathbb{E}_\sigma(\lambda\sum_{t=1}^{\infty}(1-\lambda)^{t-1} g^i(a_t))$. λ is the discount rate.

σ is an *uniform equilibrium* if:

1. $\forall \epsilon > 0$, σ is a ϵ-equilibrium under average payoff when played long enough.

2. $((\gamma^i_T(\sigma))_{i\in N})_T$ has a limit in $\mathbb{R}^N$ as $T \to \infty$.

Remark 1. the smaller λ is, the more patient the players are.

Their equilibrium sets are clearly nested.

Proposition 1. *$E_1 \subset E_T \subset E_\infty$ and $E_1 \subset E_\lambda \subset E_\infty$.*

Next, we introduce the concepts of feasible payoffs and rationality.

Definition 1.6.

- The vector payoff function $g : S \to R^n, g(s) = (g_1(s^1, \cdots, g_n(s^n)))$, the feasible payoff set is $F = co(g(S)) = g(\Delta(S)), E_\infty \subset F$.

- Punishment level of player i is $v^i = \underset{x^{-i}}{min}\, \underset{x^i}{max}\, g^i(x^i, x^{-i})$, no matter which strategy i uses, other players always have a way to punish i by making him gain no more than v^i.

- Individually rational payoff set is $IR = \{u = (u^i) | u^i \geq v^i, \forall i \in N\}$.

Conversely, i can always gain no less than v_i, so we assume that if a player is rational, his payoff should be in IR, thus $E = IR \cap F$ is the largest payoff set in this setting.

2 Folk Theorems

2.1 The Folk Theorem

Theorem 3. *everything is possible if you are willing to play forever: $E_\infty = E$.*

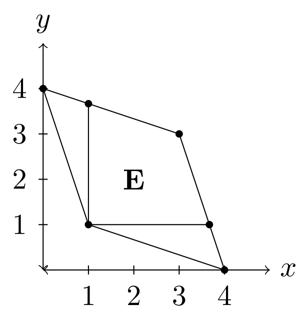
We sketch a proof below.

Proof. We need to show that $E \subset E_\infty$, take $x \in E$, there is a *path* u to the payoff. All players start by playing the action prescribed by u and continue to do so until someone deviates. If player i deviates, all other players switch to the strategies that can punish i to make him gain less than v_i forever. The one-stage gain from deviation contributes 0 to the total payoff of player i in the average payoff. The utility of a deviating player cannot be higher than his punishment level. Hence by employing punishment, all players stay on the intended path which is easily verified in E_∞. □

One natural question: should we expect $E_T \to E_\infty$ in Hausdorff distance? No, the above Folk theorem is clearly too ideal and is hardly ever the practical case, the prisoner's dilemma is a counterexample, we see clearly below that E is much larger than any E_T (just a fixed point).

Recall the definition of E_∞, the path σ need not be a Nash equilibrium up to any stage, namely, the players need to be willing to lose a little for cooperation, which can lead to long-term greater profit.

Individually rational and feasible set in the prisoner's dilemma



This again shows that **Nash equilibrium ≠ optimal profile**, willing to lose by ϵ can lead to cooperation if all players agree to follow the path σ, but in real games, we have to take into account how much one trust each other.

2.2 Other Versions

There are many more versions of Folk theorems depending on what kind of repeated games you want to study, we present them here without proving them. We use v_i to denote the punishment level.

Theorem 4. *If there are two players or $\exists u \in E, u_i > v_i$, then $E_\lambda \to E$ in Hausdorff distance between sets, as $\lambda \to 0$.*

The condition for E_T to converge to E is stricter and is false in the prisoner's dilemma.

Theorem 5. *$\exists u \in E_1, u_i > v_i$, then $E_T \to E$ in Hausdorff distance.*

Finally, this note is only an introduction to the language of game theory, and there is much more to explore by making our assumptions more practical and introducing more possibilities such as imperfect information, stochastic games, and other kinds of equilibrium.

References

[1] Rida Laraki, Jérôme Renault, and Sylvain Sorin, *Mathematical foundations of game theory*, Universitext, Springer, Cham, 2019, For the French original see [MR3135265]. MR 3967750