LCH and separable, but not σ-compact

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There is an interesting homework problem from 202B taught by Prof Michael Christ at UC Berkeley, Spring 2023.

Problem 1. *In a locally compact and Hausdorff (LCH), and second compact space, every open set is σ-compact.*

Since LCH is a weaker condition than *σ-compact*, and *separability* is weaker than *second countability*, I'm curious about the following question:

Question 1. *Is an LCH and separable topological space also σ-compact?*

Here is an example serving as a negative answer to this question.

Let $X = \mathbb{R}$, with topology defined as follows. For each $x \in \mathbb{R} \setminus \mathbb{Q}$ choose an infinite subset $S(x) \subset \mathbb{Q}$ that is bounded, and whose only accumulation point in $\mathbb{R}$ is x. A base for the topology for X consists of sets of the following two types.

- All subsets of $\mathbb{Q}$.
- all subsets of X of the form $\{x\} \cup S$, where $x \in \mathbb{R} \setminus \mathbb{Q}$, $S \subset S(x)$, and $S(x) \setminus S$ is finite.

One can check this indeed forms a topological basis.

X **is Hausdorff.** Indeed, consider any $x \neq x' \in X$. If x, x' both belong to $\mathbb{R} \setminus \mathbb{Q}$ then $A = S(x) \cap S(x')$ is finite. (If the intersection were infinite, it would have both x, x' as accumulation points). Choosing $S = S(x) \setminus S(x')$ and $S' = S(x') \setminus S(x)$, $\{x\} \cup S$ and $\{x'\} \cup S'$ are disjoint neighborhoods of x, x', respectively.

X **is locally compact.** If $x \in \mathbb{Q}$ then $\{x\}$ is a compact neighborhood of x. If $x \in \mathbb{R} \setminus \mathbb{Q}$ then $\{x\} \cup S(x)$ is a compact neighborhood of x. Indeed, any open set containing x also contains all but finitely many elements of $S(x)$, so any open cover of $\{x\} \cup S(x)$ has a finite subcover.

$X = \mathbb{R}$**, it's separable.**

X **is not σ-compact**. Any compact subset K of X contains only finite elements of $\mathbb{R} \setminus \mathbb{Q}$. Suppose to the contrary that $x_n \in (\mathbb{R} \setminus \mathbb{Q}) \cap K$ are distinct for $n \in \mathbb{N}$. Define $V_n = \{x_n\} \cup S(x_n)$. Define $V_0 = \mathbb{Q}$. The sets V_n, indexed by $\{0, 1, 2, \ldots\}$, form an open cover of K. Each contains only one element of $\mathbb{R} \setminus \mathbb{Q}$. Therefore there is no finite subcover.

Therefore the uncountable set $\mathbb{R} \setminus \mathbb{Q}$ is not contained in any countable union of compact subsets of X.

So yes, there could be σ-compact open sets in such a space, but the space itself may not be σ-compact, as shown in our example.